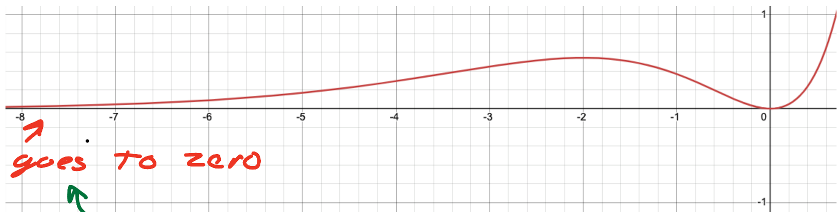
Period 3, Oct 30, 2024

Water Bottle Rocket

Students build a rocket that must meet predetermined specifications. At the Olympiad, rockets will be "fueled" with 355 milliliters of water. The rocket with the greatest combined "hang time" and patch design score will be declared the winner. Each school may enter one (1) rocket built by a team consisting of three (3) students. All teams must have: Water-Bottle Vehicle (constructed and launch-ready) and Team Patch.

$$\lim_{x\to a}\frac{F(x)}{h(x)}=\lim_{x\to a}\frac{F'(x)}{g'(x)}$$

$$\frac{F(a)}{h(a)}=\frac{\infty}{\infty}\Rightarrow\frac{0}{0}$$



goes to zero

Example 4 Evaluate each

a) $\displaystyle\lim_{x\to-\infty}\frac{x^2}{e^{-x}}$

Plug in $-\infty$

$$\frac{(-\infty)^2}{e^{-(-\infty)}}=\frac{\infty}{e^{\infty}}=\frac{\infty}{\infty}$$

$$\lim_{x\to-\infty}\frac{x^2}{e^{-x}}=\lim_{x\to-\infty}\frac{2x}{-e^{-x}}=\lim_{x\to-\infty}\frac{2}{e^{-x}}=\frac{2}{e^{-(-\infty)}}=\frac{2}{e^{\infty}}=\frac{2}{\infty}=0$$

$$\frac{2(-\infty)}{-e^{-(-\infty)}}=\frac{-\infty}{-\infty}$$

$$\frac{\sqrt{1+0}-1-\frac{0}{2}}{0^2}=\frac{\sqrt{1}-1-0}{0}=\frac{1-1}{0}=\frac{0}{0}$$

b) $\displaystyle\lim_{x\to0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}=\lim_{x\to0}\frac{(1+x)^{\frac12}-1-\frac12x}{x^2}=\lim_{x\to0}\frac{\frac12(1+x)^{-\frac12}\cdot1-0-\frac12}{2x}$

Plug in 0 $=\dfrac{\frac{1}{2\sqrt{1+0}}-0-\frac12}{2\cdot0=0}$ $\quad\frac12-\frac12\quad\frac{0}{0}$

$$\lim_{x\to0}\frac{\frac12\cdot-\frac12(1+x)^{-\frac32}\cdot1-0-0}{2}=\lim_{x\to0}\frac{\frac{-1}{4\sqrt{(1+x)^3}}}{2}\Rightarrow\frac{\frac{-1}{4\sqrt{(1+0)^3}}}{2}=\frac{-\frac14}{\frac21}=-\frac18$$

Example 6 Evaluate each of the following limits.

a) $\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{x^{\frac{1}{2}}}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \frac{\frac{1}{2\sqrt{\infty}}}{e^{\infty}} = \frac{\frac{1}{2\infty}}{\infty} = \frac{0}{\infty}$

$\Downarrow$

$\frac{\sqrt{\infty}}{e^{\infty}} = \frac{\infty}{\infty}$

$\frac{\frac{1}{2\sqrt{10{,}000}}}{e^{10{,}000}} = \frac{1}{2\sqrt{1000}} \cdot \frac{1}{e^{1500}}$ $\boxed{=0}$

b) $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{1\,(x-1)}{\ln x\,(x-1)} - \frac{\ln x \cdot 1}{\ln x\,(x-1)}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{(\ln x)(x-1)}$

$\Downarrow$

$\frac{1}{\ln 1} - \frac{1}{1-1} = \frac{1}{0} - \frac{1}{0}$

$\Downarrow$

$\frac{1-1-\ln 1}{(\ln 1)(1-1)} = \frac{0-0}{0\cdot 0} = \frac{0}{0}$

$\ln 1 = 0$

$\lim_{x\to 1^+} \frac{1-0-\frac{1}{x}}{\frac{1}{x}(x-1)+(\ln x)(1)} = \lim_{x\to 1^+} \frac{\frac{x}{x}-\frac{1}{x}}{\frac{x-1}{x}+\frac{x\ln x}{x}} = \lim_{x\to 1^+} \frac{\frac{x-1}{x}}{\frac{x-1+x\ln x}{x}} = \lim_{x\to 1^+} \frac{x-1}{\cancel{x}} \cdot \frac{\cancel{x}}{x-1+x\ln x}$

Plug in 1

$\frac{1-\frac{1}{1}}{\frac{1}{1}(1-1)(\ln 1)} = \frac{1-1}{1\cdot 0+0} = \frac{0}{0+0} = \frac{0}{0}$

$\lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x} = \lim_{x\to 1^+} \frac{1}{1-0+1\cdot\ln x + x\cdot\frac{1}{x}}$

$\lim_{x\to 1^+} \frac{1}{1+\ln x+\frac{x}{x}} = \frac{1}{1+\ln 1+\frac{1}{1}} = \frac{1}{1+0+1}$

$\boxed{\frac{1}{2}}$

b) $\lim_{x\to\infty} \frac{3x^2-1}{2x^2+1}$

$y = \lim_{x\to\infty} \frac{3x^2-1}{2x^2+1} \approx \lim_{x\to\infty} \frac{3\cancel{x^2}}{2\cancel{x^2}} = \frac{3}{2}$

$y = \lim_{x\to\infty} \frac{3x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{6\cancel{x}}{4\cancel{x}} = \frac{6}{4} = \frac{3}{2}$

$\frac{3(\infty)^2-1}{2(\infty)^2+1} = \frac{\infty}{\infty}$

$\lim_{x\to a} \frac{F(x)}{h(x)} = \lim_{x\to a} \frac{F'(x)}{h'(x)}$

$\frac{F(a)}{h(a)} = \frac{\infty}{\infty}$ or $\frac{0}{0}$

$\frac{\frac{\ln x}{1}}{\frac{1}{x}} = \frac{\ln x}{1} \cdot \frac{x}{1} = \frac{x\ln x}{1} = x\ln x$

b) $\lim_{x\to 0^+} x^x \implies y = \lim_{x\to 0^+} x^x \Rightarrow \ln y = \ln\left[\lim_{x\to 0^+} x^x\right] = \lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x\ln x$

⇓
Plug in 0 0^0

$$\ln y = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-1\cdot x^{-2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} \frac{1}{x}\cdot\frac{-x^2}{1} = -x$$

Plug in 0 ⇓

$\frac{\ln 0}{\frac{1}{0}} = \frac{-\infty}{\infty}$

$-0 = 0$

$\Rightarrow \ln y = 0 \Leftrightarrow e^0 = y$

$1 = y$

c) $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

$$\lim_{x\to 7} \frac{(\sqrt{x+2}-3)(\sqrt{x+2}+3)}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{x+2-3\cancel{\sqrt{x+2}}+\cancel{3\sqrt{x+2}}-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{\cancel{(x-7)}}{\cancel{(x-7)}(\sqrt{x+2}+3)}$$

$\frac{\sqrt{7+2}-3}{7-7} = \frac{\sqrt{9}-3}{7-7} = \frac{3-3}{7-7} = \frac{0}{0}$

$\lim_{x\to 7} \frac{1}{\sqrt{x+2}+3}$

$\frac{1}{\sqrt{7+2}+3} = \frac{1}{\sqrt{9}+3}$

$\frac{1}{3+3} = \frac{1}{6}$

$$\lim_{x\to 7} \frac{(x+2)^{\frac{1}{2}}-3}{x-7} = \lim_{x\to 7} \frac{\frac{1}{2}(x+2)^{-\frac{1}{2}}\cdot 1-0}{1-0} = \lim_{x\to 7} \frac{\frac{1}{2\sqrt{x+2}}}{1}$$

$\frac{1}{2\sqrt{7+2}} = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}$

a) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ $\Rightarrow$ $y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x \Rightarrow \ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x\right]$

$$y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$$

$$\left(1+\frac{2}{\infty}\right)^\infty = 1^\infty$$

$$\left(1+\frac{2}{1000}\right)^{1000} = (1.002)^{1000}$$

$$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x$$

$$\ln y = \lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right)$$

Plug in ∞

$$\infty \cdot \ln\left(1+\frac{2}{\infty}\right) = \infty \cdot \ln 1 = \infty \cdot 0$$

$$\ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x}+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln\left(\frac{x+2}{x}\right)}{\frac{1}{x}}$$

$$\ln y = \lim_{x\to\infty} \frac{\ln(x+2) - \ln x}{x^{-1}}$$

$$\ln y = \lim_{x\to\infty} \frac{\frac{1x}{(x+2)x} - \frac{1(x+2)}{x(x+2)}}{-1x^{-2}}$$

$$\ln y = \lim_{x\to\infty} \frac{\frac{\cancel{x}-\cancel{x}-2}{x(x+2)}}{\frac{-1}{x^2}} = \lim_{x\to 0} \frac{+2}{\cancel{x}(x+2)} \cdot \frac{x^{\cancel{2}}}{+1}$$

$$\ln y = \lim_{x\to\infty} \frac{2x}{x+2} \approx \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}} = 2$$

$$\ln y = 2$$

$$y = e^2 = 7.387$$

1.

$$5x^3+11x^3y^2+2y^3=4x+7y^4$$

$$\cancel{15x^2}+\cancel{33x^2\cdot y^2}+11x^3\cdot 2y\frac{dy}{dx}+6y^2\frac{dy}{dx}=4+\cancel{28y^3\frac{dy}{dx}}\quad -15x^2-33x^2y^2$$

$$\cancel{-15x^2}\;\cancel{-33x^2y^2}\qquad -28y^3\frac{dy}{dx}\qquad \cancel{-28y^3\frac{dy}{dx}}$$

$$22x^3y\frac{dy}{dx}+6y^2\frac{dy}{dx}-28y^3\frac{dy}{dx}=-15x^2-33x^2y^2+4$$

$$\frac{\frac{dy}{dx}\cancel{(22x^3y+6y^2-28y^3)}}{\cancel{22x^3y+6y^2-28y^3}}=\frac{-15x^2-33x^2y^2+4}{22x^3y+6y^2-28y^3}$$

$$\frac{dy}{dx}=\frac{-15x^2-33x^2y+4}{22x^3y+6y^2-28y^3}=\frac{15x^2+33x^2y-4}{-22x^3y-6y^2+28y^3}$$

$$4x^2+6x^2y^3+y^4=6x+2y^3$$

$$8x+12xy^3+6x^2\cdot 3y^2\frac{dy}{dx}+4y^3\frac{dy}{dx}=6+6y^2\frac{dy}{dx}$$

$$\cancel{8x}+\cancel{12xy^3}+18x^2y^2\frac{dy}{dx}+4y^3\frac{dy}{dx}=6+6y^2\frac{dy}{dx}\quad -8x-12xy^3$$

$$\cancel{-8x}\;\cancel{-12xy^3}\qquad -6y^2\frac{dy}{dx}\qquad -6y^2\frac{dy}{dx}$$

$$18x^2y^2\frac{dy}{dx}+4y^3\frac{dy}{dx}-6y^2\frac{dy}{dx}=6-8x-12xy^3$$

$$\frac{\frac{dy}{dx}(18x^2y^2+4y^3-6y^2)}{18x^2y^2+4y^3-6y^2}=\frac{6-8x-12y^3}{18x^2y^2+4y^3-6y^2}=\frac{6-8x-12xy^3}{18x^2y^2+4y^3-6y^2}=\frac{8x+12xy^3-6}{6y^2-18x^2y^2-4y^3}$$

$$=\frac{4x+6xy^3-3}{3y^2-9x^2y^2-2y^3}$$

$$4x^2y+3y^7=6$$

$$\cancel{8xy}+4x^2\cdot\frac{dy}{dx}+21y^6\frac{dy}{dx}=0-8xy$$
$$\cancel{-8xy}$$

$$\frac{dy}{dx}\frac{\cancel{(4x^2+21y^6)}}{\cancel{4x^2+21y^6}}=\frac{-8xy}{4x^2+21y^6}$$

$$\frac{dy}{dx}=\frac{-8xy}{4x^2+21y^6}=\frac{8xy}{-4x^2-21y^6}=-\frac{8xy}{4x^2+21y^6}$$

$$\frac{d^2y}{dx^2}=\frac{[-8y+-8x\frac{dy}{dx}](4x^2+21y^6)-(-8xy)[8x+126y^5\frac{dy}{dx}]}{(4x^2+21y^6)^2}$$

$$\frac{d^2y}{dx^2}=\frac{[-8y-8x(\frac{-8xy}{4x^2+21y^6})](4x^2+21y^6)+8xy[8x+126y^5(\frac{-8xy}{4x^2+21y^6})]}{(4x^2+21y^6)^2}$$

$$\frac{d^2y}{dx^2}=\frac{-8y(4x^2+21y^6)-8x(\frac{-8xy}{\cancel{4x^2+21y^6}})(\cancel{4x^2+21y^6})+64x^2y+1008xy^6\cdot\frac{-8xy}{4x^2+21y^6}}{(4x^2+21y^6)^2}$$

$$\frac{d^2y}{dx}=\frac{-32x^2y-168y^7+64x^2y+64x^2y-\frac{8064x^2y^7}{4x^2+21y^6}}{(4x^2+21y^6)^2}$$

$$3xy^2 + 4y^3 = 6$$

$$\cancel{3 \cdot y^2} + 3x \cdot 2y\frac{dy}{dx} + 12y^2\frac{dy}{dx} = 0$$

$$-3y^2 \qquad\qquad -3y^2$$

$$\frac{\frac{dy}{dx}\left(\cancel{6xy+12y^2}\right)}{\cancel{6xy+12y^2}} = \frac{-3y^2}{6xy+12y^2}$$

$$\frac{dy}{dx} = \frac{-3y^2}{6xy+12y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-6y\frac{dy}{dx}(6xy+12y^2) - (-3y^2)\left[6 \cdot y + 6x \cdot \frac{dy}{dx} + 24y\frac{dy}{dx}\right]}{(6xy+12y^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-6y\left(\frac{-3y^2}{\cancel{6xy+12y^2}}\right)\left(\cancel{6xy+12y^2}\right) + 3y^2\left[6y + 6x\left(\frac{-3y^2}{6xy+12y^2}\right) + 24y\left(\frac{-3y^2}{6xy+12y^2}\right)\right]}{(6xy+12y^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{18y^3 + 18y^3 + 18xy^2\left(\frac{-3y^2}{6xy+12y^2}\right) + 24y\left(\frac{-3y^2}{6xy+12y^2}\right)}{(6xy+12y^2)^2}$$

or

$$\frac{dy}{dx} = \frac{-3y^2}{6xy+12y^2} = \frac{3y(-y)}{3y(x+4y)} = \frac{-y}{x+4y}$$

$$\frac{d^2y}{dx} = \frac{-\frac{dy}{dx}(x+4y) - (-y(1+4\frac{dy}{dx}))}{(x+4y)^2}$$

$$\frac{-\cancel{\left(\frac{-y}{x+4y}\right)} \cdot \cancel{(x+4y)} + y\left(1+4\left(\frac{-y}{x+4y}\right)\right)}{(x+4y)^2} = \frac{y+y-\frac{4y^2}{x+4y}}{(x+4y)^2}$$

$$\frac{\frac{(x+4y)2y}{(x+4y)1} - \frac{4y^2}{x+4y}}{(x+4y)^2} = \frac{\frac{2xy+8y^2-4y^2}{x+4y}}{\frac{(x+4y)^2}{1}} = \frac{2xy+4y^2}{(x+4y)} \cdot \frac{1}{x+4y}$$

$$\frac{d^2y}{dx} = \frac{(2xy+4y^2)}{(x+4y)^3}$$